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NON-INTEGER TOPOLOGICAL NUMBER AND INFINITENESS OF ACTION *

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We give a simple proof for the non-existence of finite action, instanton-like Euclidean field configurations with fractional topological number in 4-dimensional, SU(2) Yang-Mills theory.

1. Introduction

The topology of gauge field configurations [1] has important consequences in the quantization of the theory [2,3]. It is normally assumed that topological numbers are integers, although some proposals concerning the possible relevance of fractional topological numbers have been made [4–6].

It was proved, by means of the Atiyah-Singer theorem [7], that, if the fields are compactifiable, the topological number is necessarily an integer [8]. The compactification method, described in [9], requires a regularity condition on the fields at infinity, which is not identical to one obtained from the more physical requirement that the action should be finite In two-dimensional quantum electrodynamics (Schwinger model [10]), finite action half-integer field configurations do play an important role [4,5] in the confining aspects of this model. It is, therefore, worthwhile to understand in more detail, to what extend does the finite action requirement preclude the existence of a fractional topological number in four-dimensional SU(2) gauge theory.

In this note, in sect. 2, we will consider one-valued field configurations which tend asymptotically to a singular gauge transformation and carry a fractional topological number **. In sect. 3, we give a general proof that those configurations

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** The multivalued configurations of ref. [11] do not seem appropriate in the simply connected Euclidean space-time.

have necessarily infinite action, thereby exhibiting the basic clash between finiteness of action and non-integer topological numbers.

We will always work in a Euclidean space-time.

2. Non-integer topological number

Since it is a prerequisite for having finite action, we will only consider field configurations which are pure gauge at infinity i. e.

$$A_\mu \xrightarrow[x\to\infty]{} g^{-1}\partial_\mu g \equiv T_\mu \,, \tag{2.1}$$

where $g(x)$ are the gauge-group (SU(2)) matrices.

It is well known [1], that when

$$g(x) = g_1(x) = \frac{x_4 - i\boldsymbol{x}\cdot\boldsymbol{\sigma}}{\sqrt{x_4^2 + \boldsymbol{x}^2}} \,, \tag{2.2}$$

the topological number is one. It also follows from [1], that if $g(x)$ is regular on a big sphere, the topological number is always an integer. There is, however, *a priori* no reason why we should not consider singular (say at the south pole) gauge transformations provided we can guarantee local integrability of the action density. In the Schwinger model, for instance, the finite action half-integer configuration of [5], has its asypmtotic behaviour controlled by the singular gauge transformation

$$g_{1/2}(x) = \exp(i\phi/2) \,, \tag{2.3}$$

with ϕ the polar angle, $0 \leqslant \phi < 2\pi$, whereas the analog of (2.2) is given by the regular (outside of the origin) gauge transformation

$$g_1(x) = \mathrm{e}^{i\phi} \,. \tag{2.4}$$

By rewriting (2.2) as

$$g_1(x) = \exp(-i\chi\hat{\boldsymbol{x}}\cdot\boldsymbol{\sigma}) \tag{2.5}$$

with

$$x_4 = R\cos\chi \,, \qquad |\boldsymbol{x}| = R\sin\chi \,, \qquad \hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \,,$$

one can, in perfect analogy to the two-dimensional case, introduce

$$g_a(x) = \exp(-ia\chi\boldsymbol{x}\cdot\boldsymbol{\sigma}/|\boldsymbol{x}|) = [g_1(x)]^a \,. \tag{2.6}$$

The topological quantum number of a regular field configuration, whose asymptotic behaviour is given by (2.1) with g given by g_a in (2.6), can be computed from

$$q = -\frac{1}{24\pi^2}\,\mathrm{Tr}\int_0^\pi \mathrm{d}\chi \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\; \epsilon^{ijk} T_i^a T_j^a T_k^a \,, \tag{2.7}$$

where

$$T_i^a = g_a^{-1}\partial_i g_a\ , \qquad i=\chi,\theta,\phi\ , \tag{2.8}$$
$$\epsilon^{\chi\theta\phi} = 1\ ,$$

and χ, θ, ϕ are the angular coordinates in four dimensions.

Using (2.8), one readily finds

$$\mathrm{Tr}\,\epsilon^{ijk} T_i^a T_j^a T_k^a = -\,12a\,\sin^2 a\chi\,\sin\theta$$

and with (2.7)

$$q = a - \frac{\sin 2\pi a}{2\pi}\ . \tag{2.9}$$

Whenever a is an integer or half-integer, we have q equal to a. It is amusing to know that only in those cases we have additivity of the topological number.

To summarize, we have exhibited configurations with an arbitrary, not necessarily integer, topological number.

3. Infiniteness of action

In this section we shall prove that the action is infinite if the topological number is non-integer.

Writing the Yang-Mills field action

$$S = -\tfrac{1}{2}\,\mathrm{Tr}\int F_{\mu\nu}F_{\mu\nu}\,\mathrm{d}^4x$$

is spherical coordinates, one finds [12]

$$S = -\,\mathrm{Tr}\int_0^\pi \mathrm{d}\chi \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \int_0^\infty \mathrm{d}R \left[\frac{F_{R\chi}^2}{R^2} + \frac{F_{R\theta}^2}{R^2\sin^2\chi} + \frac{F_{R\phi}^2}{R^2\sin^2\chi\sin^2\theta}\right.$$
$$\left. + \frac{F_{\chi\theta}^2}{R^4\sin^2\chi} + \frac{F_{\chi\phi}^2}{R^4\sin^2\chi\sin^2\theta} + \frac{F_{\theta\phi}^2}{R^4\sin^4\chi\sin^2\theta}\right] R^3 \sin^2\chi\,\sin\theta\ , \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\ , \qquad \mu,\nu = \chi,\theta,\phi,R \tag{3.2}$$

is the field intensity tensor in spherical coordinates.

Let us consider field potentials of the form

$$A_\mu = f(R,\chi)\, g_a^{-1}\partial_\mu g_a = f(R,\chi)\, T_\mu^a \tag{3.3}$$

with f a general function, chosen to satisfy

$$f(0,\chi) = 0\ , \tag{3.4a}$$

$$f(R,\pi) = 0 \,, \tag{3.4b}$$

$$f(\infty,\chi) = 1 \,, \qquad \chi \neq \pi \,. \tag{3.4c}$$

Condition (3.4a) ensures that the potential is regular at the origin. In order to compensate for the singularity of g_a (for non-integer a's) on the $\chi = \pi$ semiaxis, we have to impose condition (3.4b). Finally, condition (3.4c) is necessary to have the potential behaving asymptotically as pure gauge.

With the above conditions, it is always possible to have the action density locally integrable.

As we showed in the last section, the regular instanton-like field configuration (3.3) has a topological number given by (2.9).

Consider now the 2nd, 3rd and 6th terms of the action, respectively

$$A = -\,\mathrm{Tr} \int F^2_{R\theta} R \sin\theta \; \mathrm{d}\chi \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}R \,, \tag{3.5a}$$

$$B = -\,\mathrm{Tr} \int \frac{F^2_{R\phi}}{\sin\theta} R \; \mathrm{d}\chi \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}R \,, \tag{3.5b}$$

$$C = -\,\mathrm{Tr} \int \frac{F^2_{\theta\phi}}{R \sin^2\chi \sin\theta} \; \mathrm{d}\chi \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}R \,. \tag{3.5c}$$

Inserting (3.3) into (3.2) and using the following properties of T^a_i,

$$T^{a2}_\chi = -a^2, \qquad T^{a2}_\theta = -\sin^2 a\chi, \qquad T^{a2}_\phi = -\sin^2 a\chi \sin^2\theta \,,$$

$$\{T^a_i, T^a_j\} = 0 \,, \qquad i \neq j \,,$$

we get (for details cf. [12])

$$A = B = 8\pi \int_0^\pi \mathrm{d}\chi \sin^2 a\chi \int_0^\infty \mathrm{d}R \, R \, [\partial_R f]^2 \,, \tag{3.6}$$

$$C = 32\pi \int_0^\pi \frac{\sin^4 a\chi}{\sin^2\chi} \mathrm{d}\chi \int_0^\infty \frac{[f^2 - f]^2}{R} \mathrm{d}R \,. \tag{3.7}$$

Calling

$$b(\chi) = \int_0^\infty R \, [\partial_R f]^2 \, \mathrm{d}R \,, \tag{3.8}$$

$$h(\chi) = \int_0^\infty \frac{[f^2 - f]^2}{R} \, \mathrm{d}R \,, \tag{3.9}$$

we find, by Schwartz inequality

$$b(\chi)\, h(\chi) \geqslant \left| \int_0^\infty (\partial_R f)[f^2 - f]\, \mathrm{d}R \right|^2 . \tag{3.10}$$

Introducing a new integration variable $\nu = f$, we get, for $\chi \neq \pi$

$$b(\chi) h(\chi) \geqslant \left| \int_0^1 (\nu^2 - \nu)\, \mathrm{d}\nu \right|^2 = \tfrac{1}{36} . \tag{3.11}$$

Noting that $b(\chi)$ and $h(\chi)$ are positive semidefinite, we obtain, applying Schwartz inequality once more

$$AC = BC = 256\pi^2 \int_0^\pi \sin^2 a\chi\, b(\chi)\, \mathrm{d}\chi \int_0^\pi \frac{\sin^4 a\chi}{\sin^2\chi} h(\chi)\, \mathrm{d}\chi$$

$$\geqslant 256\pi^2 \left| \int_0^\pi \frac{\sin^3 a\chi}{\sin \chi} \sqrt{b(\chi)\, h(\chi)}\, \mathrm{d}\chi \right|^2$$

By virtue of (3.11), we get

$$AC = BC \geqslant \frac{64\pi^2}{9} \left| \int_0^\pi \frac{\sin^3 a\chi}{\sin \chi} \mathrm{d}\chi \right|^2 = \begin{cases} \infty \text{ if } a \text{ is non-integer} \\ \text{finite if } a \text{ is integer} \end{cases} .$$

We see, therefore, that with a non-integer topological number, either C or A and B will have to diverge, so that the action will always be infinite.

4. Conclusion

We have seen that instanton-like configurations that carry a non-integer topological number have necessarily infinite action. Cancelation of divergences, in different terms of the action is impossible since they are all positive definite.

The choice (3.3) with (3.4) makes the action density locally integrable. The divergence of the action appears only after the integration is carried out over whole space-time.

A more general (θ and ϕ) functional dependence of the function f, introduced in sect. 3, although not necessary to make the action density locally integrable, would be introduced without changing our results.

Although we have not considered the most general configuration with a non-integer topological number, we believe that our considerations exhibit the basic clash between finiteness of action and non-integrality of the topological charge.

Since we have considered pure Yang-Mills theory, we can not *a priori* exclude the possibility envisaged in [13], that the effective action, obtained after integration of the fermion variables, in a theory of quarks interacting with gluons, could be finite for non-integer topological charge.

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